Lecture 9 : Basic $\mathcal{L}^2$ Convergence Theorem

STAT205 *Lecturer: Jim Pitman* Scribe: *Peng Li* <pengli@berkeley.edu>

(This note is a revision of the work of Vinod Prabhakaran from 2002.)

9.1 Basic $\mathcal{L}^2$ Convergence Theorem

Theorem 9.1 (Basic $\mathcal{L}^2$ Convergence Theorem) *Let X_1 X_2, ... be independent random variables with $\mathbb{E}(X_i) = 0$ and $\mathbb{E}(X_i^2) = \sigma_i^2 < \infty$, $i = 1, 2, \ldots$, and $S_n = X_1 + X_2 + \cdots + X_n$. If $\sum_{i=1}^{\infty} \sigma_i^2 < \infty$, then S_n converges a.s. and in $\mathcal{L}^2$ to some S_∞ with $\mathbb{E}(S_\infty^2) = \sum_{i=1}^{\infty} \sigma_i^2$.*

Recall: We have done this before, with the conclusion for the $\mathcal{L}^2$ case with the weaker assumption that $\mathbb{E}(X_i X_j) = 0$ for $i \neq j$. The only new thing is the conclusion of a.s. convergence for the independent case. In fact, the proof just uses Kolmogorov's inequality from the last lecture. Thus the conclusion is valid for a martingale $\{S_n\}$ with $\mathbb{E}[X_{n+1} f(X_1, \ldots, X_n)] = 0$ for all bounded measurable $f : \mathbb{R}^n \to \mathbb{R}$.

Proof: First note that $\mathcal{L}^2$ convergence and existence of S_∞ is implied by the orthogonality of the X_i's: since $\mathbb{E}(X_i X_j) = 0$ for $i \neq j$,

$$
\begin{aligned}
\mathbb{E}(S_n^2) &= \sum_{i=1}^{n} \sigma_i^2, \text{ and} \\
\mathbb{E}((S_n - S_m)^2) &= \sum_{i=m+1}^{n} \sigma_i^2 \;\to\; 0 \text{ as } m, n \;\to\; \infty,
\end{aligned}
$$

so S_n is Cauchy in $\mathcal{L}^2$. Since $\mathcal{L}^2$ is complete, there is a unique S_∞ (up to a.s. equivalence) such that $S_n \;\to\; S_\infty$ in $\mathcal{L}^2$.

Turning to a.s. convergence, the method is to show the sequence (S_n) is a.s. Cauchy. The limit of S_n then exists a.s. by completeness of the set of real numbers. The same argument applies more generally to martingale differences X_i. Note that this method gives S_∞ more explicitly, and does not appeal to completeness of $\mathcal{L}^2$.

Recall that S_n is Cauchy a.s. means $M_n := \sup_{p,q \geq n} |S_p - S_q| \;\to\; 0$ a.s. Note that $0 \leq M_n(\omega) \downarrow$ implies that $M_n(\omega)$ converges to a limit in $[0, \infty]$. So, if $\mathbb{P}(M_n > \epsilon) \;\to\; 0$ for all $\epsilon > 0$, then $M_n \downarrow 0$ a.s.

Let $M_n^* := \sup_{p\geq n}|S_p - S_n|$. By the triangle inequality,

$$|S_p - S_q| \leq |S_p - S_n| + |S_q - S_n| \;\Rightarrow\; M_n^* \leq M_n \leq 2M_n^*,$$

so it is sufficient to show that $M_n^* \xrightarrow{P} 0$.

For all $\epsilon > 0$,

$$\begin{aligned}\mathbb{P}\left(\sup_{p\geq n}|S_p - S_n| > \epsilon\right) &= \lim_{N\to\infty}\mathbb{P}\left(\max_{n\leq p\leq N}|S_p - S_n| > \epsilon\right)\\ &\leq \lim_{N\to\infty}\sum_{i=n+1}^{N}\frac{\sigma_i^2}{\epsilon^2} = \sum_{i=n+1}^{\infty}\frac{\sigma_i^2}{\epsilon^2}\end{aligned}$$

where we applied Kolmogorov's inequality in the second step. Since $\sum_{i=1}^{\infty}\sigma_i^2 < \infty$,

$$\lim_{n\to\infty}\mathbb{P}\left(\sup_{p\leq n}|S_p - S_n| > \epsilon\right) = 0$$

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Remark: Just orthogonality rather than independence of the X_is is not enough to get an a.s. limit. Counterexamples are hard. According to classical results of Rademacher-Menchoff, for orthogonal X_i the condition $\sum_i(\log^2 i)\sigma_i^2 < \infty$ is enough for a.s. convergence of S_n, whereas if $b_i \uparrow$ with $b_i = o(\log^2 i)$ there exist orthogonal X_i such that $\sum_i b_i\sigma_i^2 < \infty$ and S_n diverges almost surely.

9.2 Kolmogorov's Three-Series Theorem

An easy consequence of the Basic $\mathcal{L}^2$ Convergence Theorem is the sufficiency part of Kolmogorov's three-series theorem:

Theorem 9.2 (Kolmogorov) *Let $X_1, X_2, \ldots$ be independent. Fix $b > 0$. Convergence of the following three series*

- $\sum_n \mathbb{P}(|X_n| > b) < \infty$
- $\sum_n \mathbb{E}(X_n\mathbf{1}_{(|X_n|<b)})$ *converges to a finite limit*
- $\sum_n \mathbf{V}ar(X_n\mathbf{1}_{(|X_n|<b)}) < \infty$

is equivalent to $\mathbb{P}(\sum_n X_n$ *converges to a finite limit*$) = 1$

Note: If any one of the three series diverges then

$$\mathbb{P}\left(\sum_n X_n \text{ converges to a finite limit}\right) = 0$$

by Kolmogorov's zero-one law (will be shown later). Note also that if one or more of the series diverges for some b, then one or more of the series must diverge for every b, but exactly which of the three series diverge may depend on b. Examples can be given of 8 possible combinations of convergence/divergence.

Proof: [Proof of sufficiency] That is, convergence of all 3 series implies $\sum_n X_n$ converges a.s. Let $X_n' = X_n 1_{(|X_n| \leq b)}$. Since $\sum_n \mathbb{P}(X_n' \neq X_n) = \sum_n \mathbb{P}(|X_n| > b) < \infty$, the Borel-Cantelli lemma gives $\mathbb{P}(X_n' \neq X_n \text{ i.o.}) = 0$ which implies $\mathbb{P}(X_n' = X_n \text{ ev.}) = 1$. Also if $X_n'(\omega) = X_n(\omega)$ ev., then $\sum_n X_n(\omega)$ converges $\Leftrightarrow \sum_n X_n'(\omega)$ converges.

Therefore it is enough to show that

$$\mathbb{P}\left(\sum_n X_n' \text{ converges to a finite limit}\right) = 1$$

Now

$$\sum_{n=1}^{N} X_n' = \sum_{n=1}^{N} (X_n' - \mathbb{E}(X_n')) \; + \; \sum_{n=1}^{N} \mathbb{E}(X_n').$$

$\sum_{n=1}^{N} \mathbb{E}(X_n')$ has a limit as $N \to \infty$ by hypothesis, and

$$\sum_{n=1}^{\infty} \mathbb{E}((X_n' - \mathbb{E}(X_n'))^2) = \sum_{n=1}^{\infty} \text{var}(X_n') < \infty$$

implies that $\sum_{n=1}^{\infty} (X_n' - \mathbb{E}(X_n'))$ converges a.s. by the basic $\mathcal{L}^2$ convergence theorem. ∎

9.3 Kolmogorov's 0-1 Law

$X_1, X_2, \ldots$ are independent random variables (not necessarily real valued). Let $\mathcal{F}_n' = \sigma(X_n, X_{n+1}, \ldots)$ = the future after time n = the smallest σ-field with respect to which all the $X_m, m \geq n$ are measurable. Let $\mathcal{T} = \cap_n \mathcal{F}_n'$ = the remote future, or *tail* σ−field.

Example 9.3 $\{\omega : S_n(\omega) \text{ } \textit{converges}\} \in \mathcal{T}$.

Theorem 9.4 (Kolmogorov's 0-1 Law) *If $X_1, X_2, \ldots$ are independent and $A \in \mathcal{T}$ then $\mathbb{P}(A) = 0$ or 1.*

Proof: The idea is to show that A is independent of itself, that is, $\mathbb{P}(A \cap A) = \mathbb{P}(A)\mathbb{P}(A)$, so $\mathbb{P}(A) = \mathbb{P}(A)^2$, and hence $\mathbb{P}(A) = 0$ or 1. We will prove this in two steps:

(a) $A \in \sigma(X_1, \ldots, X_k)$ and $B \in \sigma(X_{k+1}, X_{k+2}, \ldots)$ are independent.

Proof of (a): If $B \in \sigma(X_{k+1}, \ldots, X_{k+j})$ for some j, this follows from (4.5) in chapter 1 of [1]. Since $\sigma(X_1, \ldots, X_k)$ and $\cup_j \sigma(X_{k+1}, \ldots, X_{k+j})$ are π-systems that contains Ω (a) follows from (4.5) in chapter 1 of [1]).

(b) $A \in \sigma(X_1, X_2, \ldots)$ and $B \in \mathcal{T}$ are independent.

Proof of (b): Since $\mathcal{T} \subset \sigma(X_{k+1}, X_{k+2}, \ldots)$, if $A \in \sigma(X_1, \ldots, X_k)$ for some k, this follows from (a). $\cup_k \sigma(X_1, \ldots, X_k)$ and $\mathcal{T}$ are π-systems that contain Ω, so (b) follows from (4.2) in chapter 1 of [1].

Since $\mathcal{T} \subset \sigma(X_1, X_2, \ldots)$, (b) implies that $A \in \mathcal{T}$ is independent of itself and the theorem follows.

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Recall *Kronecker's lemma*: If $a_n \uparrow \infty$ and $\sum_{n=1}^{\infty} X_n/a_n$ converges a.s., then $(\sum_{m=1}^{n} X_m)/a_n \xrightarrow{a.s.} 0$.

Let $X_1, X_2, \ldots$ be independent with mean 0 and $S_n = X_1 + X_2 + \cdots + X_n$. If $\sum_{n=1}^{\infty} \mathbb{E}(X_n^2)/a_n^2 < \infty$, then by the basic $\mathcal{L}^2$ convergence theorem $\sum_{n=1}^{\infty} X_n/a_n$ converges a.s. Then $S_n/a_n \to 0$ a.s.

Example 9.5 *Let $X_1, X_2, \ldots$ be i.i.d., $\mathbb{E}(X_i) = 0$, and $\mathbb{E}(X_i^2) = \sigma^2 < \infty$.*
Take $a_n = n$:

$$\sum_{n=1}^{\infty} \frac{\sigma^2}{n^2} < \infty \Rightarrow \frac{S_n}{n} \xrightarrow{a.s.} 0.$$

Now take $a_n = n^{\frac{1}{2}+\epsilon}$, $\epsilon > 0$:

$$\sum_{n=1}^{\infty} \frac{\sigma^2}{n^{1+2\epsilon}} < \infty \Rightarrow \frac{S_n}{n^{\frac{1}{2}+\epsilon}} \xrightarrow{a.s.} 0.$$

References

[1] Richard Durrett. *Probability: theory and examples, 3rd edition.* Thomson Brooks/Cole, 2005.